

# INTRODUCTION TO DYNAMICAL SYSTEMS

## Solutions Problem Set 4

**Exercise 1.** Assume that  $\phi, \Phi$  are as in the structural stability theorem in lecture3.pdf. Also assume with

$$\Phi(x) = 2x + \hat{\psi}(x), \quad \hat{\psi}(x+1) = \hat{\psi}(x),$$

that  $\hat{\psi} \in C^1(\mathbb{R})$ , and finally that  $\hat{\psi}(0) = 0, \hat{\psi}'(0) \neq 0$ . Then show that there is no open interval  $(a, b) \subset \mathbb{R}$  on which the function  $u$  which solves

$$\Phi(u(x)) = u(2x) \tag{1}$$

is of class  $C^1$ . *Hint: first show that if  $u$  is  $C^1$  on some open interval, then it is already  $C^1$  everywhere. Then show that there is a dense set of points where the derivative  $u'$  vanishes.*

*Solution.* Notice that  $u$  solving the fixed point problem (1) means that for  $x \in (a, b)$  where  $u$  is differentiable, we have

$$u'(2x) = u'(x) \left( 1 + \frac{\hat{\psi}'(u(x))}{2} \right). \tag{2}$$

Since  $u \in C^1((a, b))$  and  $\hat{\psi} \in C^1(\mathbb{R})$ , we find that  $u \in C^1((2a, 2b))$ , and by iteration,

$$u \in C^1 \left( \bigcup_{n \geq 0} (2^n a, 2^n b) \right).$$

Finally, notice that there must be  $N \geq 0$  for which  $2^N(b-a) > 1$ , and this means that  $u$  is differentiable in  $(2^N a, 2^N b)$ , which contains an interval of length 1, and by periodicity  $u$  is differentiable everywhere.

To show that  $u'$  vanishes in a dense set, notice that since  $\hat{\psi}$  is Lipschitz with  $L \in [0, 1)$ , the rightmost factor on the right-hand side of (2) is never equal to zero, and therefore

$$u'(x) = 0 \text{ if and only if } u'(2x) = 0. \tag{3}$$

Moreover, by looking at (2) at  $x = 0$  one can use this same argument to show that  $u'(0) = 0$ . Differentiating  $u(x+1) = u(x) + 1$  implicitly yields

$$u'(x+1) = u'(x),$$

meaning that  $u'(n) = 0$  for all  $n \in \mathbb{Z}$ . Combining this with (3) and iterating  $k \in \mathbb{N}$  times yields

$$u' \left( \frac{n}{2^k} \right) = 0.$$

Hence  $u'$  vanishes on the set  $S = \{n/2^k : n \in \mathbb{Z}, k \in \mathbb{N} \cup \{0\}\}$ , which is dense in  $\mathbb{R}$ . Therefore  $u' \equiv 0$ , which contradicts the fact that  $u$  is increasing.  $\square$

### Exercise 2.

1. Let  $T$  be a contraction on the complete metric space  $(X, d)$  with contraction factor  $\lambda \in (0, 1)$ , i.e.  $d(Tx, Ty) \leq \lambda d(x, y)$ . Then if  $x^*$  denotes the fixed point of  $T$ , show that for any  $x_1 \in X$  there is a constant  $C > 0$  such that

$$d(T^k(x_1), x^*) \leq C\lambda^k.$$

2. Let  $Tg(x) = \Phi^{-1}(g(2x))$  be as in lecture3.pdf, acting on the complete metric space  $X$  as in the lecture. By optimizing the choice of  $k$  in (i), show that there is a number  $\alpha = \alpha(L) \in (0, 1)$  and a constant  $C_* > 0$  such that

$$|u(x) - u(y)| \leq C_* |x - y|^\alpha$$

for every  $x, y \in \mathbb{R}$  with  $|x - y| \leq 1$ . Here  $u$  solves the fixed point problem (1). thus the conjugating function  $h$  is Hölder continuous.

*Solution.* For the first part, notice that

$$\begin{aligned} d(T^k x_1, x^*) &= \lim_{n \rightarrow \infty} d(T^k x_1, T^n x_1) \leq \lambda^k \sup_{n \geq 1} d(x_1, T^n x_1) \\ &\leq \lambda^k \sup_{n \geq 1} \sum_{j=1}^n d(T^{j-1} x_1, T^j x_1) \leq \lambda^k \sum_{j \geq 1} \lambda^j d(x_1, T x_1) = \frac{1}{1 - \lambda} d(x_1, T x_1) \lambda^k. \end{aligned}$$

It suffices to set  $C = d(x_1, T x_1)/(1 - \lambda)$ . For the second part, coming back to the proof in the lectures, we notice that if we let  $g$  be the identity (as a starting point for the iteration), then  $T^k g(x) = \Phi^{-k}(2^k x)$ , which implies

$$|T^k(g(x)) - T^k(g(y))| \leq 2^k r_1^{-k} \cdot |x - y|.$$

Using the first part of the exercise, we conclude that

$$|u(x) - u(y)| \leq 2d(u, T^k g) + |T^k(g(x)) - T^k(g(y))| \leq 2C r_1^{-k} + 2^k r_1^{-k} \cdot |x - y|.$$

Now, let  $\alpha > 0$  and choose  $k$  to be the smallest natural number such that

$$2^k \geq \frac{C}{|x - y|}, \quad 2^\alpha = r_1,$$

to find

$$|u(x) - u(y)| \leq 4 \cdot 2^{k-1} |x - y| r_1^{-k} \leq 4C \left( \frac{|x - y|}{C} \right)^\alpha \leq D |x - y|^\alpha.$$

Finally, since  $r_1 = 2 - L$ , notice that we may explicitly write

$$\alpha = \frac{\log(2 - L)}{\log 2}.$$

□

**Exercise 3.** Let  $X$  a finite measure space and  $T: X \rightarrow X$  measure preserving. For measurable sets  $A, B \subset X$ , write

$$A \Delta B := (A \setminus B) \cup (B \setminus A).$$

Show that  $T$  is ergodic if and only if

$$m(A \Delta T^{-1}(A)) = 0$$

implies that  $m(A) = 0$  or  $m(A) = m(X)$ . *Hint:*  $\bigcap_{n \geq 1} \bigcup_{j \geq n} T^{-j}(A)$ .

*Solution.* Notice that the *only if* direction is direct. In particular, if  $A \subset X$  is invariant by  $T$ , then the symmetric difference of  $A$  and  $T^{-1}(A)$  is zero, and  $T$  is automatically ergodic.

For the *if* direction, let  $A \subset X$  be such that the condition of the statement is satisfied. We let

$$A^+ = \bigcap_{n \geq 1} \bigcup_{j \geq n} T^{-j}(A).$$

Then, it is immediate that  $A^+$  is invariant by  $T$ , and

$$m(T^{-j}(A) \Delta T^{-(j+1)}(A)) = m(T^{-j}(A \Delta T^{-1}(A))) = m(A \Delta T^{-1}(A)) = 0,$$

which readily implies that for any  $k, \ell \in \mathbb{N}$ ,

$$\begin{aligned} m(T^{-k}(A) \Delta T^{-(k+\ell)}(A)) &= m(T^{-k}(A \Delta T^{-\ell}(A))) \\ &= m(A \Delta T^{-\ell}(A)) \leq \sum_{j=0}^{\ell-1} m(T^{-j}(A) \Delta T^{-(j+1)}(A)) = 0. \end{aligned}$$

In particular, we achieve

$$T^{-k}(A) \cup T^{-(k+\ell)}(A) = T^{-k}(A) \sqcup S_{k\ell}, \quad m(S_{k\ell}) = 0.$$

Collecting all the preimages,

$$A^+ = \bigcap_{n \geq 1} \bigcup_{j \geq n} T^{-j}(A) = \bigcap_{n \geq 1} \left[ T^{-n}(A) \sqcup \bigcup_{\ell \geq 1} S_{n\ell} \right] = N \sqcup \bigcap_{n \geq 1} T^{-n}(A),$$

where  $m(N) = 0$ . Through the continuity of the measure and thanks to a simple induction argument we conclude that

$$m\left(\bigcap_{j \geq 1} T^{-j}(A)\right) = m(A),$$

and therefore  $m(A) = m(A^+)$ . Hence either  $m(A) = m(X)$  or  $m(A) = 0$ , thanks to the ergodicity of  $T$ .  $\square$

**Exercise 4.** Let  $(X, m)$  be a finite measure space and  $T: X \rightarrow X$  a measure preserving map. Using the Von Neumann Mean Ergodic Theorem, show that if  $f \in L^1(X, m)$ , then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j(x))$$

exists in  $L^1(X, m)$ , and is a  $T$ -invariant function.

*Solution.* We begin by taking  $A > 0$  and letting

$$f_A(x) = f(x) \cdot \mathbb{1}_{\{|f| \leq A\}}(x) \in L^2(X, m).$$

Then, notice that  $f_A \in L^2(X, m)$  and  $f_A$  converges to  $f$  in  $L^1(X, m)$  as  $A \rightarrow \infty$ ,

$$\|f - f_A\|_{L^1} = \|f \cdot \mathbb{1}_{\{|f| > A\}}\|_{L^1} \xrightarrow{A \rightarrow \infty} 0.$$

We choose  $\varepsilon > 0$  and let  $A > 0$  large enough so that  $\|f - f_A\|_{L^1} < \varepsilon/3$ , and when we apply the theorem to  $f_A \in L^2(X, m)$  we find that

$$\frac{1}{N} \sum_{j=0}^{N-1} f_A \circ T^j \xrightarrow{N \rightarrow \infty} f_A^*,$$

in the sense of  $L^2(X, m)$ , where  $f_A^*$  is invariant by  $T$ . Since this converges, it is in particular a Cauchy sequence, and we can estimate

$$\left\| \frac{1}{N} \sum_{j=0}^{N-1} f_A \circ T^j - \frac{1}{M} \sum_{j=0}^{M-1} f_A \circ T^j \right\|_{L^1} \leq \left\| \frac{1}{N} \sum_{j=0}^{N-1} f_A \circ T^j - \frac{1}{M} \sum_{j=0}^{M-1} f_A \circ T^j \right\|_{L^2} \sqrt{m(X)} < \varepsilon/3$$

for all  $N, M$  sufficiently big, using the Cauchy-Schwarz inequality together with the fact that  $m(X) < \infty$ . Now, using the triangle inequality we find

$$\left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^j - \frac{1}{M} \sum_{j=0}^{M-1} f \circ T^j \right\|_{L^1} < \varepsilon,$$

if  $N, M$  are sufficiently large. Therefore  $\frac{1}{N} \sum_{j=0}^{N-1} f \circ T^j$  is a Cauchy sequence in  $L^1(X, m)$ . Since this is a Banach space, there is  $f^* \in L^1(X, m)$  such that  $\frac{1}{N} \sum_{j=0}^{N-1} f \circ T^j \rightarrow f^*$ . It remains to check that  $f^*$  is invariant under  $T$ . To this end, notice that

$$\begin{aligned} \|f^* \circ T - f^*\|_{L^1} &= \lim_{N \rightarrow \infty} \left\| \frac{f \circ T^N - f}{N} \right\|_{L^1} \\ &\leq \lim_{N \rightarrow \infty} \frac{\|f \circ T^N\|_{L^1} + \|f\|_{L^1}}{N} = \lim_{N \rightarrow \infty} \frac{2\|f\|_{L^1}}{N} = 0. \end{aligned}$$

□